

Models of Finance

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- Aim : overview of the mains models of finance
 - standard arbitrage models
 - but also microstructure (behaviour) models
- material : handout
- other books :
 - modèles mathématiques de la finance : Demange Rochet Economica
 - research papers

- Chapter 1 – Introduction
 - 1. what is finance
 - 2. Assets
 - 3. Functioning of trading
 - 4. Two first models : risk neutral valuation
 - 5. Application : predictive markets
- Chapter 2 – Static model: arbitrage free condition
 - 1. Intuition
 - 2. Mathematical stuff
 - 3. No arbitrage condition in a static model
- Chapter 3 – Dynamics (finite discrete models)
 - 1. The tree of states of nature
 - 2. Stochastic process on a tree
 - 3. No arbitrage condition on a dynamic model
 - 4. Risk neutral probability

- Chapitre 4 – Continuous models
 - 1. Deterministic continuous model : the differential equation
 - 2. Brownian motion and Stochastic integral
 - 3. Arbitrage free equation
 - 4. Continuous asset valuation
- Chapter 5 – Microstructure and behaviour models
 - 1. The market efficiency hypothesis
 - 2. The Competitive Rational Expectation Equilibrium
 - 3. Bid ask spread
 - 4. Information and High frequency trading

Introduction

- Financial assets :
 - transfer wealth through time or,
 - alternatively or simultaneously, to exchange risk or mitigate it.
- Demand for financial assets is hence triggered by a need of risk coverage or transfer wealth over time. REAL POSITION
- The Fébooupa Island...Oups
- But some other agents intervene on this market although they have no initial real position.
 - Their only role is to make the market liquid : to be sure that any demand meets an offer.
 - They are called “market makers”, or specialists...or pure speculators : they post “prices”

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Types of models

- the arbitrage models (perfect markets) : risk neutral valuation
- the behaviour models : endogeneize prices, (perfect and imperfect markets).

The different models can be static, dynamic, discrete and continuous.

- The question of information (insiders vs outsiders), market efficiency...
- Other questions : bubbles...

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Assets

| Asset Class | Instrument type | | |
|-----------------------|-------------------------|----------------------------------|-----------------------------|
| | Securities | Exchange-trade derivatives | OTC derivatives |
| Debt (LT > 1year) | Bonds | Bond options or futures | Interest rate swap, options |
| Debt (ST ≤ 1year) | Bills, Commercial paper | Short term interest rate futures | forward rate agreements |
| Equity | Stock | Stock options, equity futures | stock options, exotic |
| FX (foreign exchange) | - | futures | swaps |

Definitions and notations

Ω is a “probabilizable space” whose elements ω are the state of nature

- For a stock:

Definition

(Notation) A stock is associated to equity and hence is an asset which gives right to “random” dividends $d(t, \omega) = \tilde{d}(t)$ at time t . Stocks are traded on an Exchange platform. Supply and demand determine prices.

- For a bond :

Definition

(Notation) a Bond is associated to a loan issued (by a borrower) at date t_0 for a length (maturity) T . Defined by the sequence of future “sure” cash flows (for the holder) $d(t)$ (coupons) and, when T is finite, a final payment $h(t_0 + T)$, (in this case $t > T + t_0 \Rightarrow d(t) = 0$).

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Definition

- “ultimately” or “In fine” bonds are defined by a face value (nominal or principal) N and a nominal rate r_0 per period such that $d(t) = N * r_0$ for $t < t_0 + T$ and $h(t_0 + T) = N(1 + r_0)$.
- Constant annuities are bonds such that $h(t_0 + T) = d(t)$
- A perpetuity is a bond for which $T = +\infty$.
- The nominal zero-coupon at date t_0 with maturity T is bond is such that $d(t) = 0$ and $h(t_0 + T) = N = 1$

Definition

The forward contract negotiated at t_0 for a term T sets at date t_0 a forward price $f(t_0, T)$ for a given quantity of a given good. Hence if $p(t)$ is the spot price at date t of this good, the seller of the forward contract will earn $f(t_0, T) - p(T)$ at T paid by the buyer.

- Forwards : OTC
- Futures : standadized and organized : at all t $f(t-1, T) - f(t, T)$ is paid by the buyer to the seller through the clearinghouse.

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Options

Definition

An option is a contract between two parties and allows one of the parties to ensure, on payment of a premium, the right (but not the obligation) to buy or sell to the other party a particular asset at a predetermined price at the end of a certain period (called European options) or during a certain period (American options).

underlying asset :

- financial asset (stock, bond, treasury bond, futures, currencies, indices, etc..)
- physical asset (agricultural or mineral).
- The value of the option is the amount of the premium that the option buyer has to pay to the seller. An option is said to be negotiable if it can be traded on a regulated market. Otherwise, one speaks of OTC trading.

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Example of an european option

Definition

Call (on a stock) : the right to to buy a stock (spot price $S(t)$) at the price K at time T : the cash flow at time T is hence

$$\max \{0, S(T) - K\} = (S(T) - K)^+$$

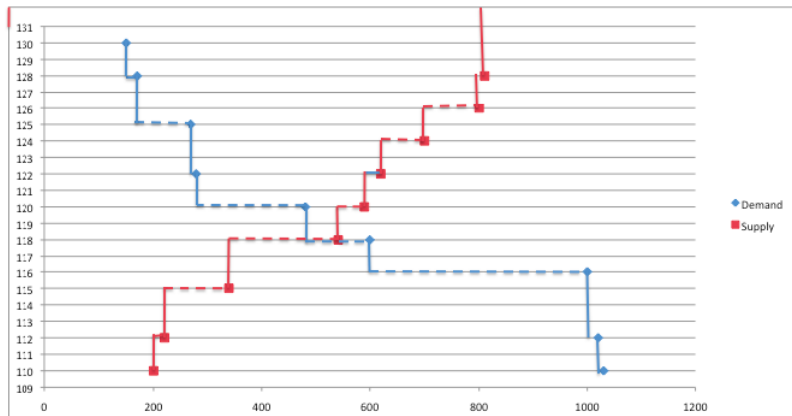
Order book : example

| Cumulative | Quantity | Price | Price | Quantity | Cumulative |
|------------|----------|----------|----------|----------|------------|
| 100 | 100 | no limit | no limit | 150 | 150 |
| 150 | 50 | 130 | 110 | 50 | 200 |
| 170 | 20 | 128 | 112 | 20 | 220 |
| 270 | 100 | 125 | 115 | 120 | 340 |
| 280 | 10 | 122 | 118 | 200 | 540 |
| 480 | 200 | 120 | 120 | 50 | 590 |
| 600 | 120 | 118 | 122 | 30 | 620 |
| 1000 | 400 | 116 | 124 | 80 | 700 |
| 1020 | 20 | 112 | 126 | 100 | 800 |
| 1030 | 10 | 110 | 128 | 10 | 810 |

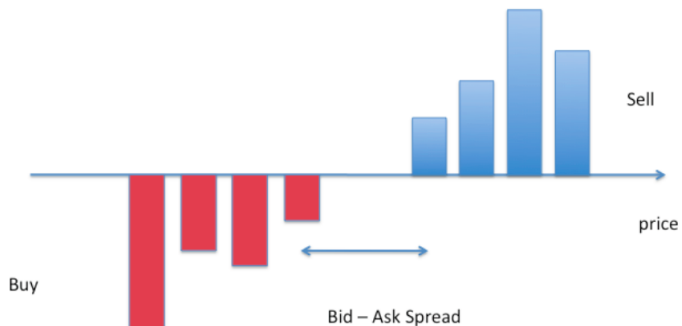
Equilibrium 118

| Cumulative | Quantity | Price | Price | Quantity | Cumulative |
|------------|----------|-------|-------|----------|------------|
| 60 | 60 | 118 | 120 | 50 | 50 |
| 460 | 400 | 116 | 122 | 30 | 80 |
| 480 | 20 | 112 | 124 | 80 | 160 |
| 490 | 10 | 110 | 126 | 100 | 260 |
| | | | 128 | 10 | 270 |

Graph



Another way



Intraday

Two first models

- A simple Risk Neutral Valuation Model
 - A simple behaviour model : the risk neutral probability

Risk Neutral Valuation

- Two bonds

- The first one issued by a government. Maturity $T = 1$ year, nominal, $N = 100$ euros, price is $B_1 = 90$.

- This bond has hence a yield r_1 such that :

$$1 + r_1 = \frac{100}{90} = \frac{10}{9} = 1,111\dots$$

or

$$r_1 = 11,1\%$$

- The second one is a corporate bond. Repay is also 100 euro in one year.

- but there is a risk of default : repay will be 80.

- Its price B_2 cannot take any value: this bond gives at least 80 and at most 100 so that its price is such that :

$$80 \leq (1 + r_1) B_2 \leq 100$$

or

$$\frac{90}{100} 80 = 72 \leq B_2 \leq 90$$

which means that the “nominal yield” r_2 of this bond is larger than r_1

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- This inequality will be referred, in the sequel, as the “arbitrage free” or

Risk Neutral Valuation

- Intuitively i: B_2 close to 72 means that there is a “high probability” of default. Conversely B_2 close to 90 means a low probability of default.
- One can define this probability \hat{Q} by the following reasoning
 - B_2 gives $(1 + r_1)B_2 = \frac{10}{9}B_2$ when invested in the sure bond
 - When invested on the risky one, the expected revenue will be $\hat{Q}80 + (1 - \hat{Q})100 = 100 - 20\hat{Q}$
 - So that we could write : $100 - 20\hat{Q} = \frac{10}{9}B_2$ or $B_2 = 90 - 18\hat{Q}$
- Remark that $0 \leq \hat{Q} \leq 1 \iff 90 \geq B_2 \geq 72$
- For instance : $\hat{Q} = \frac{1}{2} \iff B_2 = 81$, $\hat{Q} = \frac{1}{3} \iff B_2 = 84$
 $\hat{Q} = \frac{1}{6} \iff B_2 = 87$ $\hat{Q} = \frac{1}{9} \iff B_2 = 88$

- on top of these two bonds, a new financial product :
 - CDS (credit default swap) : promises 1 euro in case of default of the firm, price : C
- portfolio strategy :
 - buy the second bond and immunate risk by buying 20 CDS.
 - This strategy gives exactly 100 euros in one year,
- portfolio completely identical with the bond 1.
 - The price of this portfolio must then be equal to B_1 (because the market is arbitrage free)

$$B_2 + 20C = B_1$$

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- This can be rewritten :

$$C = \frac{90 - B_2}{20} = \frac{90 - 90 + 18\hat{Q}}{20} = \frac{90}{100}\hat{Q}$$

or

$$C = \frac{90}{100} [\hat{Q} \times 1 + (1 - \hat{Q}) \times 0]$$

- This is the expected discounted value under the probability \hat{Q}
- For all the assets we have that the price is equal to the expected discounted value :

$$B_1 = \frac{90}{100} [\hat{Q} \times 100 + (1 - \hat{Q}) \times 100] = \frac{\hat{Q} \times 100 + (1 - \hat{Q}) \times 100}{1 + r_1}$$

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Theorem

Under that probability \hat{Q} , the value of any asset (existing or composite) is simply its expected present value

- Mathematically : two states of nature, no default of firm , default of firm
- the repay in one year can be described by a two components vector
 - For the risk-free asset $d_1 = (100, 100)$
 - For the second bond $d_2 = (100, 80)$
 - and for the CDS $d_{CDS} = (0, 1)$
- Clearly these 3 vectors are not independent since one of them is a linear combination of the two other :

$$d_{CDS} = \frac{1}{20}d_1 - \frac{1}{20}d_2$$

We deduce that the price :

$$\text{Price of } (d_{CDS}) = \frac{1}{20}\text{Price of } (d_1) - \frac{1}{20}\text{Price of } (d_2)$$

That is :

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- Let the Matrix

$$D \equiv \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 100 & 100 \\ 100 & 80 \end{bmatrix}$$

- A portfolio $\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$ gives the payments :

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} = {}^t \theta D = [\theta_1 \times 100 + \theta_2 \times 100, \theta_1 \times 100 + \theta_2 \times 80]$$

Here the matrix is invertible, so that if I want payments $\begin{bmatrix} v_1 & v_2 \end{bmatrix}$ I need

to buy the portfolio $\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = [{}^t D^{-1}] \cdot {}^t v$

With

$$D^{-1} = \frac{1}{20} \begin{bmatrix} -\frac{80}{100} & 1 \\ 1 & -1 \end{bmatrix}$$

- So I have to pay : $B_1 \theta_1 + B_2 \theta_2 = {}^t \theta B = v D^{-1} B = v Q$

with

$$Q = \begin{bmatrix} \frac{1}{20} (B_2 - \frac{80}{100} B_1) \\ \frac{1}{20} (B_1 - B_2) \end{bmatrix}$$

- We say that the market is complete in the sense that any asset whose repay is $d = (v_1, v_2)$ can be obtained as a portfolio of the two bonds:

$$d = \frac{1}{20} \left(v_2 - \frac{80}{100} v_1 \right) d_1 + \frac{1}{20} (v_1 - v_2) d_2$$

or :

$$d = \frac{1}{20} \left(d_2 - \frac{80}{100} d_1 \right) v_1 + \frac{d_1 - d_2}{20} v_2$$

- The “arbitrage free “ hypothesis implies that the price of this new asset is simply the same linear combination of the prices of the two bonds :

$$P(d) = \frac{1}{20} \left(B_2 - \frac{80}{100} B_1 \right) v_1 + \frac{B_1 - B_2}{20} v_2$$

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$$\frac{1}{20} \left(B_2 - \frac{80}{100} B_1 \right) + \frac{B_1 - B_2}{20} = \frac{B_1}{100} = \frac{1}{1+r}$$

- So that ,

$$(1+r) \frac{1}{20} \left(B_2 - \frac{80}{100} B_1 \right), \quad (1+r) \frac{B_1 - B_2}{20}$$

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Second model

- there are only two individuals A and B.
- A : entrepreneur, random income . There are two “states of nature” (events) :
- the good state H where the income is large W_H
 - the bad state L where it is low $W_L < W_H$.
 - B has a constant income w .
 - In this economy the total income is hence $W_H + w$ or $W_L + w$.

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Second model

- A has a real initial risky position, he would like to lower it : get a final income X such that X_H (final income when H) and X_L (final income when L) are close together.
- trade with B by selling her some share (stocks) of his firm.
 - If he sells a share $\alpha \in [0,1]$ at a price S , final situation
 - A will have $X_L = \alpha S + (1 - \alpha)W_L$ and $X_H = \alpha S + (1 - \alpha)W_H$
 - And B $Y_L = w - \alpha S + \alpha W_L$ and $Y_H = w - \alpha S + \alpha W_H$
 - So that the risk beared by A will be reduced to
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- Compute the expected incomes :

$$\widehat{X} = \pi_H X_H + \pi_L X_L = \alpha S + (1 - \alpha) \widehat{W}$$

$$\widehat{Y} = \pi_H Y_H + \pi_L Y_L = w - \alpha S + \alpha \widehat{W}$$

- For a given price, S , we define the supply $\alpha^S(S)$ being the share that A is ready to offer, and the demand $\alpha^D(S)$, as the share, B is ready to buy at this price.

- Compute the expected incomes :

$$\widehat{X} = \pi_H X_H + \pi_L X_L = \alpha S + (1 - \alpha) \widehat{W}$$

$$\widehat{Y} = \pi_H Y_H + \pi_L Y_L = w - \alpha S + \alpha \widehat{W}$$

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$$S = \widehat{W} = \pi_H W_H + \pi_L W_L$$

meaning that the price of the firm is its expected income?

- In that case the expected income is unchanged for both, \widehat{W} for A and w for B, whatever α !
- there is a “pure” exchange of risk : $X_H - X_L = (1 - \alpha)(W_H - W_L)$,
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Definition

risk aversion : among all risky incomes having the same expectation, a risk averse decision maker always prefers the one with no risk at all

With price $S = \widehat{W}$, Among all the possible α , A prefers $\alpha = 1$ and B prefers $\alpha = 0$ The supply is not equal to the demand!

Second model

- Intuitively S must be lower than \widehat{W} in order to be acceptable by B. But to what extent?
- In order to study that problem, we make the expected utility assumption.

Definition

For any decision maker i , there exists a real increasing function u_i such that, for any risky incomes \tilde{x} and \tilde{y} , i prefers \tilde{x} to \tilde{y} , if and only if :

$$\mathbb{E}u_i(\tilde{x}) \geq \mathbb{E}u_i(\tilde{y})$$

Theorem

in the framework of the expected utility hypothesis, i is risk averse if and only if u_i is concave

Second model

Supply :

$$U_A(S, \alpha) = \pi_H u_A(\alpha S + (1 - \alpha)W_H) + \pi_L u_A(\alpha S + (1 - \alpha)W_L)$$

$$\pi_H u'_A(X_H)(S - W_H) + \pi_L u'_A(X_L)(S - W_L) = 0$$

The same reasoning for B :

$$\pi_H u'_B(Y_H)(W_H - S) + \pi_L u'_B(Y_L)(W_L - S) = 0$$

$$S = \pi_H \frac{u'_A(X_H)}{\pi_H u'_A(X_H) + \pi_L u'_A(X_L)} W_H + \pi_L \frac{u'_A(X_L)}{\pi_H u'_A(X_H) + \pi_L u'_A(X_L)} W_L$$

Application : predictive markets

- Iowa university predictive markets : Here
- Let us play : One bond “Red” (10 euros if a red card is drawn from this card set : here
 - no initial position
 - with initial position

Summary

- The **first main message** of your talk in one or two lines.
 - The **second main message** of your talk in one or two lines.
 - Perhaps a **third message**, but not more than that.
-
- Outlook
 - What we have not done yet.
 - Even more stuff.