Models of Finance

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Models of finance

- Aim : overview of the mains models of finance
  - standard arbitrage models
  - but also microstructure (behaviour) models
- material : handout
- other books :
  - modèles mathématiques de la finance : Demange Rochet Economica
  - research papers

Synopsis

- Chapter 1 – Introduction
  - 1. what is finance
  - 2. Assets
  - 3. Functioning of trading
  - 4. Two first models of RNV : risk neutral valuation for a CDS and predictive markets
  - 5. A simple behaviour model
- Chapter 2 – Static model: arbitrage free condition
  - 1. Intuition
  - 2. Mathematical stuff
  - 3. No arbitrage condition in a static model
- Chapter 3 – Dynamics (finite discrete models
  - 1. The tree of states of nature
  - 2. Stochastic process on a tree
  - 3. No arbitrage condition on a dynamic model
  - 4. Risk neutral probability

Synopsis

- Chapter 4 – Continuous models
  - 1. Deterministic continuous model : the differential equation
  - 2. Brownian motion and Stochastic integral
  - 3. Arbitrage free equation
  - 4. Continuous asset valuation
- Chapter 5 – Microstructure and behaviour models
  - 1. The market efficiency hypothesis
  - 2. The Competitive Rational Expectation Equilibrium
  - 3. Bid ask spread
  - 4. Information and High frequency trading

Introduction

- Financial assets :
  - transfer wealth through time or,
  - alternatively or simultaneously, to exchange risk or mitigate it.
- Demand for financial assets is hence triggered by a need of risk coverage or transfer wealth over time. REAL POSITION
- The Fébooupa Island...Oups
- But some other agents intervene on this market although they have no initial real position.
  - Their only role is to make the market liquid : to be sure that any demand meets an offer.
    - They are called “market makers”, or specialists...or pure speculators : they post “prices”

Types of models

- the arbitrage models (perfect markets) : risk neutral valuation
- the behaviour models : endogeneize prices, (perfect and imperfect markets).

The different models can be static, dynamic, discrete and continuous.

- The question of information (insiders vs outsiders), market efficiency...
  - Other questions : bubbles...

Assets

Asset Class	Instrument type		
	Securities	Exchange-trade derivatives	OTC derivatives
Debt (LT>1year)	Bonds	Bond options or futures	Interest rate swap, options
Debt (ST≤1year)	Bills, Commercial paper	Short term interest rate futures	forward rate agreements
Equity	Stock	Stock options, equity futures	stock options, exotic
FX (foreign exchange)	-	futures	swaps

Definitions and notations

Ω is a "probabilizable space" whose elements ω are the state of nature

- For a stock:

Definition

(Notation) A stock is associated to equity and hence is an asset which gives right to "random" dividends $d(t,\omega)$ at time t. Stocks are traded on an Exchange platform. Supply an demand determine prices.

- For a bond :

Definition

(Notation) a Bond is associated to a loan issued (by a borrower) at date t_0 for a length (maturity) T. Defined by the sequence of "sure" cash flows (for the holder) $d(t)$, $t = t_0 + 1, \ldots t_0 + T - 1$ (coupons) and, when T is finite, a final payment $d(t_0 + T)$, (in this case $t > T + t_0 \Rightarrow d(t) = 0$) .

Bonds

Definition

- "ultimately" or "In fine" bonds are defined by a face value (nominal or principal) $N$and a nominal rate r_0 per period such that $d(t) = N * r_0$ for $t < t_0 + T$ and $d(t_0 + T) = N(1 + r_0)$.
- Constant annuities are bonds such that $d(t) \equiv d$
- A perpetuity is a bond for which $T = +\infty$.
- The nominal zero-coupon at date t_0 with maturity T is bond is such that $d(t) = 0$ and $d(t_0 + T) = N = 1$

Forwards and futures

Definition

The forward contract negociated at t for a term T sets at date t a forward price $f(t,T)$ for a given quantity of a given good. Hence if $p(T) = f(T,T)$ is the spot price at date T of this good, the seller of the forward contract will earn $f(t,T) - p(T)$ at T payed by the buyer.

- Forwards : OTC
- Futures : standadized and organized : at all s $t+1 \leq s \leq T$, $f(s-1,T) - f(s,T)$ is payed by the buyer to the seller through the clearinghouse, so that the buyer pays

$$\sum_{s=t+1}^{T} (f(s-1,T) - f(s,T)) = \sum_{s=t}^{T-1} f(s,T) - \sum_{s=t+1}^{T} f(s,T)$$

$$= f(t,T) - f(T,T)$$

Options

Definition

An option is a contract between two parties and allows one of the parties to ensure, on payment of a premium, the right (but not the obligation) to buy or sell to the other party a particular asset at a predetermined price at the end of a certain period (called European options) or during a certain period (American options). underlying asset :

- financial asset (stock, bond, treasury bond, futures, currencies, indices, etc..)
- physical asset (agricultural or mineral).
- The value of the option is the amount of the premium that the option buyer has to pay to the seller. An option is said to be negotiable if it can be traded on a regulated market. Otherwise, one speaks of OTC trading.

Example of an european option

Definition

Call (on a stock) : the right to to buy a stock (spot price $S(t)$) at the price K at time T: the cash flow at time T is hence $\max\{0, S(T) - K\} = (S(T) - K)^+$

Trading

Order book : example

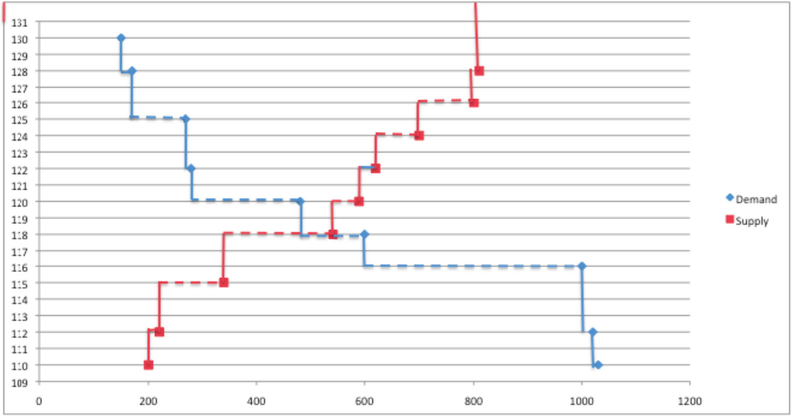
Cumulative	Quantity	Price	Price	Quantity	Cumulative
100	100	no limit	no limit	150	150
150	50	130	110	50	200
170	20	128	112	20	220
270	100	125	115	120	340
280	10	122	118	200	540
480	200	120	120	50	590
600	120	118	122	30	620
1000	400	116	124	80	700
1020	20	112	126	100	800
1030	10	110	128	10	810

Trading

Equilibrium 118

Cumulative	Quantity	Price	Price	Quantity	Cumulative
60	60	118	120	50	50
460	400	116	122	30	80
480	20	112	124	80	160
490	10	110	126	100	260
			128	10	270

Graph



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111
110
109
0
200
400
600
800
1000
1200
Demand
Supply

Another way

Intraday

Two first models

- A simple Risk Neutral Valuation Model
    - A simple behaviour model : the risk neutral probability

Risk Neutral Valuation

Theorem

Equilibrium supply/demand (and hence arbitrage free condition) implies that there exists a probability measure (risk neutral measure, market percieved probability...) such that under that probability, the expected return of any asset is equal to the risk free interest rate.

Risk Neutral Valuation

- Two bonds
  - The first one issued by a government. Maturity $T = 1$ year, nominal, $N = 100$ euros, price is $B_1 = 90$.
    - This bond has hence a yield r_1 such that :

$$1 + r_1 = \frac{100}{90} = \frac{10}{9} = 1,111...$$

or

$$r_1 = 11,1\%$$

- The second one is a corporate bond. Repay is also 100 euro in one year.
  - but there is a risk of default : repay will be 80.
    - Its price B_2 cannot take any value: this bond gives at least 80 and at most 100 so that its price is such that :

$$80 \leq (1 + r_1) B_2 \leq 100$$

or

$$\frac{90}{100} 80 = 72 \leq B_2 \leq 90$$

- This inequality will be refered, in the sequel, as the "arbitrage free" or "absence of arbitrage" condition.

Risk Neutral Valuation

- Intuitively : B_2 close to 72 means that there is a "high probability" of default. Conversely B_2 close to 90 means a low probability of default.
- One can define this probability $\widehat{Q}$ by the following reasonning
  - B_2 gives $(1+r_1)B_2 = \frac{10}{9}B_2$ when invested in the risk free bond
    - When invested on the risky one, the expected revenue will be $\widehat{Q}80 + \left(1-\widehat{Q}\right)100 = 100 - 20\widehat{Q}$
    - So that we could write : $100 - 20\widehat{Q} = \frac{10}{9}B_2 \text{ or } B_2 = 90 - 18\widehat{Q}$
  - Remark that $0 \leq \widehat{Q} \leq 1 \iff 90 \geq B_2 \geq 72$
  - For instance : $\widehat{Q} = \frac{1}{2} \iff B_2 = 81$, $\widehat{Q} = \frac{1}{3} \iff B_2 = 84$ $\widehat{Q} = \frac{1}{6} \iff B_2 = 87, \widehat{Q} = \frac{1}{9} \iff B_2 = 88$

RNV

- on top of these two bonds, a new financial product :
  - CDS (credit default swap) : promises 1 euro in case of default of the firm, price : C
- portfolio strategy :
  - buy the second bond and immunate risk by buying 20 CDS.
    - This strategy gives exactly 100 euros in one year,
- portfolio completely identical with the bond 1.
  - The price of this portfolio must then be equal to $B1$ (because the market is arbitrage free)

$$B_2 + 20C = B_1$$

- This gives the price of the CDS :

$$C = \frac{90 - B_2}{20}$$

RNV

- This can be rewritten :

$$C = \frac{90 - B_2}{20} = \frac{90 - 90 + 18\widehat{Q}}{20} = \frac{90}{100}\widehat{Q}$$

or

$$C = \frac{90}{100}\left[\widehat{Q} \times 1 + \left(1 - \widehat{Q}\right) \times 0\right]$$

- This is the expected discounted value under the probability $\widehat{Q}$
  - For all the assets we have that the price is equal to the expected discounted value :

$$B_1 = \frac{90}{100}\left[\widehat{Q} \times 100 + (1 - \widehat{Q}) \times 100\right] = \frac{\widehat{Q} \times 100 + (1 - \widehat{Q}) \times 100}{1 + r_1}$$

$$B_2 = \frac{90}{100}\left[\widehat{Q} \times 80 + (1 - \widehat{Q}) \times 100\right] = \frac{\widehat{Q} \times 80 + (1 - \widehat{Q}) \times 100}{1 + r_1}$$

$$C = \frac{90}{100}\left[\widehat{Q} \times 1 + (1 - \widehat{Q}) \times 0\right] = \frac{\widehat{Q} \times 1 + (1 - \widehat{Q}) \times 0}{1 + r_1}$$

RNV

Theorem

There exists a probability distribution $\widehat{Q}, 1-\widehat{Q}$. Under that probability, the value (price) of any asset (existing or composite) is simply its expected present (i.e. discounted) value.

or in other words:

Theorem

Equilibrium supply/demand (and hence arbitrage free condition) implies that there exists a probability measure (risk neutral measure or market percieved probability...) such that under that probability, the expected return of any asset is equal to the risk free interest rate.

RNV

- Mathematically : two states of nature, no default of firm , default of firm
- the repay in one year can be described by a two components vector
  - For the risk-free asset $d_1 = (100, 100)$
    - For the second bond $d_2 = (100, 80)$
    - and for the CDS $d_{CDS} = (0, 1)$
  - Clearly these 3 vectors are not independent since one of them is a linear combination of the two other :

$$d_{CDS} = \frac{1}{20} d_1 - \frac{1}{20} d_2$$

- We deduce that the price :

$$\text{Price of } (d_{CDS}) = \frac{1}{20} \text{Price of } (d_1) - \frac{1}{20} \text{Price of } (d_2)$$

That is :

$$C = \frac{1}{20} B_1 - \frac{1}{20} B_2$$

RNV

- Let the Matrix

$$D \equiv \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} 100 & 100 \\ 100 & 80 \end{bmatrix}$$

- A portfolio $\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$ gives the payments :

  $\begin{bmatrix} v_1 & v_2 \end{bmatrix} =^t \theta D = [\theta_1 \times 100 + \theta_2 \times 100, \theta_1 \times 100 + \theta_2 \times 80]$

  Here the matrix is invertible, so that if I want payments $\begin{bmatrix} v_1 & v_2 \end{bmatrix}$ I need to buy the portfolio $\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \left[{}^t D^{-1}\right] . {}^t v$

  With

$$D^{-1} = \frac{1}{20} \begin{bmatrix} -\frac{80}{100} & 1 \\ 1 & -1 \end{bmatrix}$$

- So I have to pay : $B_1\theta_1 + B_2\theta_2 =^t \theta B = vD^{-1}B = vQ$

  with

$$Q = \begin{bmatrix} \frac{1}{20}\left(B_2 - \frac{80}{100}B_1\right) \\ \frac{1}{20}\left(B_1 - B_2\right) \end{bmatrix}$$

RNV

- We say that the market is complete in the sense that any asset whose repay is $d = (v_1, v_2)$ can be obtained as a portfolio of the two bonds:

$$d = \frac{1}{20}\left(v_2 - \frac{80}{100}v_1\right)d_1 + \frac{1}{20}(v_1 - v_2)\,d_2$$

- or :

$$d = \frac{1}{20}\left(d_2 - \frac{80}{100}d_1\right)v_1 + \frac{d_1 - d_2}{20}v_2$$

  - The “arbitrage free “ hypothesis implies that the price of this new asset is simply the same linear combination of the prices of the two bonds :

$$P(d) = \frac{1}{20}\left(B_2 - \frac{80}{100}B_1\right)v_1 + \frac{B_1 - B_2}{20}v_2$$

RNV

- The sum of the two coefficients is

$$\frac{1}{20}\left(B_2 - \frac{80}{100}B_1\right) + \frac{B_1 - B_2}{20} = \frac{B_1}{100} = \frac{1}{1+r}$$

- So that ,

$$(1+r)\frac{1}{20}\left(B_2 - \frac{80}{100}B_1\right), \quad (1+r)\frac{B_1 - B_2}{20}$$

is a probability distribution (risk neutral probability distribution)

Second model: predicitive markets

- Iowa university predictive markets : Here
- Let us play : One bond “Red” (10 euros if a red card is drawn from this card set : here
  - no initial position
  - with initial position

Behaviour model

- there are only two individuals A and B.
- A : entrepreneur, random income . There are two "states of nature" (events) :
  - the good state H where the income is large W_H
  - the bad state L where it is low $W_L < W_H$.
  - B has a constant income w.
  - In this economy the total income is hence $W_H + w$ or $W_L + w$.

Behaviour model

- A has a real initial risky position, he would like to lower it : get a final income X such that X_H (final income when H) and X_L (final income when L) are close together.
- trade with B by selling her some share (stocks) of his firm.
  - If he sells a share $\alpha \in [0,1]$ at a price S, final situation
    - A will have $X_L = \alpha S + (1-\alpha) W_L$ and $X_H = \alpha S + (1-\alpha) W_H$
    - And B $Y_L = w - \alpha S + \alpha W_L$ and $Y_H = w - \alpha S + \alpha W_H$
  - So that the risk beared by A will be reduced to :$X_H - X_L = (1-\alpha)(W_H - W_L)$

Behaviour model

- Compute the expected incomes :

$$\widehat{X} = \pi_H X_H + \pi_L X_L = \alpha S + (1-\alpha)\widehat{W}$$

$$\widehat{Y} = \pi_H Y_H + \pi_L Y_L = w - \alpha S + \alpha \widehat{W}$$

- For a given price, S, we define the supply $\alpha^s(S)$ being the share that A is ready to offer, and the demand $\alpha^d(S)$, as the share, B is ready to buy at this price.

Behaviour model

- I am asking the question what if :

$$S = \widehat{W} = \pi_H W_H + \pi_L W_L$$

  meaning that the price of the firm is its expected income?
- In that case the expected income is unchanged for both, $\widehat{W}$ for A and w for B, whatever α!
  - there is a "pure" exchange of risk : $X_H - X_L = (1-\alpha)(W_H - W_L)$, $Y_H - Y_L = \alpha(W_H - W_L)$

Behaviour model

Definition

risk aversion : among all risky incomes having the same expectation, a risk averse decision maker always prefers the one with no risk at all

With price $S = \widehat{W}$, Among all the possible α , A prefers $\alpha = 1$ and B prefers $\alpha = 0$ The supply is not equal to the demand!

Behaviour model

- Intuitively S must be lower than $\widehat{W}$ in order to be acceptable by B. But to what extent?
- In order to study that problem , we make the expected utility assumption.

Definition

For any decision maker i , there exists a real increasing function u_i such that, for any risky incomes $\widetilde{x}$ and $\widetilde{y}$, i prefers $\widetilde{x}$ to $\widetilde{y}$, if and only if :

$$\mathbb{E}u_i(\widetilde{x}) \geq \mathbb{E}u_i(\widetilde{y})$$

Theorem

in the framework of the expected utility hypothesis, i is risk averse if and only if u_i is concave

Behaviour model

Supply :

$$U_A(S,\alpha) = \pi_H u_A(\alpha S + (1-\alpha)W_H) + \pi_L u_A(\alpha S + (1-\alpha)W_L)$$

$$\pi_H u'_A(X^*_H)(S - W_H) + \pi_L u'_A(X^*_L)(S - W_L) = 0$$

where stared quantities are equilibrium quantities
The same reasonning for B :

$$\pi_H u'_B(Y^*_H)(W_H - S) + \pi_L u'_B(Y^*_L)(W_L - S) = 0$$

$$S = \pi_H \frac{u'_A(X_H)}{\pi_H u'_A(X_H) + \pi_L u'_A(X_L)} W_H + \pi_L \frac{u'_A(X_L)}{\pi_H u'_A(X_H) + \pi_L u'_A(X_L)} W_L$$

Summary

- The first main message of your talk in one or two lines.
- The second main message of your talk in one or two lines.
- Perhaps a third message, but not more than that.

- Outlook
  - What we have not done yet.
  - Even more stuff.