MODELES DE LA FINANCE

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Ce sujet comporte 3 pages

1 Exotic american call

Consider the simplified Cox-Ross-Rubinstein model on a tree with 3 dates 0,1,2. At date 1 : two states of nature (u) and (d) , at date 2 : (uu) (ud) (du) (dd). There is a risk-free asset with constant one period return equal to r. There is a Stock whose process is given :

S_0, $S_1(u) = uS_0, S_1(d) = dS_0, S_2(uu) = u^2S_0, S_2(ud) = S_2(du) = udS_0, S_2(dd) = d^2S_0$.

u and d are positive constants such that : $u > 1 + r > d > 0$

- Q1 Recall the arbitrage free condition, give the two equations relating pseudo prices $q(u)$ and $q(d)$.

For the sequel it could be more convenient to use these two equations instead of the explicit expression of pseudo prices $q(u)$ and $q(d)$...

We define an exotic american call in the following way. One can exercice it at any date t (0,1or 3) at a price equal to $\alpha^t S_0$ (i.e. at date 0 the exercise price is S_0 at date 1, αS_0 and at date 2, $\alpha^2 S_0$) where α is a positive constant. We make the following assumption :

$u > \alpha > 1 + r$ and $ud > \alpha^2$. To find the value of such a call at date 0 we follow a backward analysis : find the value at date 2 in each state and deduce the price at date 1 in each state and deduce the value at date 0.

- Q2 What are the values of the call at date 2 in each of the 4 states?
- Q3 Consider the call at date 1 in the state (u). One can keep it of exercise it. If it was optimal to keep it, what would be its value at that date? What is the value of exercising it?
- Q4 Deduce the optimal strategy in that state, and the value of the call
- Q5 Replicate the same reasonning in the state (d).
- Q6 Give the value at date 0

2 Perpetual american put

A perpetual american put belongs to the class of assets for which it is (rather) easy to find a price formula.

An american perpetual put is "the right to sell a stock at a given price K at any date". The difference with an european put is twofold : there is no expiration date (this is why it is called perpetual), and it can be exercised at any time (this is why it is american). The assumptions are the following : there is at any time a risk-free asset whose instantaneous yield is r (constant) per unit of time, the underlying Stock has a price that follows the risk neutral dynamics :

$$dS(t) = rS(t)dt + \sigma S(t)dW(t) \tag{1}$$

Given the definitions above, suppose you hold a perpetual american put . If you decide at a date $\widehat{t}$ to exercise it, you obtain at that date a cash flow equal to $K - S\left(\widehat{t}\right)$. So, the problem is the following : at a given date, observing $S(t)$, you have to take the decision wait or exercice. Intuitively, there must exist a threshold value $\underline{S}$ such that you must wait if $S(t) > \underline{S}$ and exercise at the first time S hits $\underline{S}$, so that you get $K - \underline{S}$.

The problem is to find this $\underline{S}$.

Let $P(t)$ the value of the put at time t. Obviously this value depends only on the value of the stock : $P(t) = V(S(t))$, where V is "the value function" we are trying to find.

Assume we are at a date where $S(t) > \underline{S}$. So that waiting is optimal.

- Q1 Write $dP(t)$ (using Ito lemma, and remarking that $V(.)$ does NOT depend on t).
- Q2 Why it is necessary to have (apologies to Etienne!) : $\mathbb{E}(dP(t)) \equiv DV(P(t))\,dt = rP(t)\,dt$, where the expectation is "under the risk neutral dynamics" and D the Dynkin operator under the risk neutral dynamics.

This must be true for any time such that $S(t) > \underline{S}$. So that this equation is valid for any underlying value $x = S(t)$ larger than $\underline{S}$. Replacing $S(t)$ by x gives a second order differential equation.

- Q3 Write the second order differential equation followed by $x \to V(x)$.

This is a Riccati equation whose general solution is Ax^{α}

- Q4 Solve and find α_1 (the other solution α_0 is obvious and corresponds to the stock) such that $V_A(x) = Ax^{\alpha_1}$

For the moment we don't know A. To find A it is necessary to know the value at some point.

- Q5 What must be the value for $x = \underline{S}$? (you exercise the option)
- Q6 Give the expression of $V(S)$ with respect to $\underline{S}$
- Q7 What is the value S^* of $\underline{S}$ that maximizes $V(S)$, for any S
- Q8 Conclude : value and optimal exercise

3 Bonus : Market or corporate finance? Or a problem of a bath that fills and empties. (Useful to obtain a letter of recommendation...)

Consider a corporate whose market value (the value of the all the stocks) is $S(t)$. Assume there is a risk-free asset with constant yield per unit of time r. Under the risk neutral probability we have hence (with a strong abuse of notation) $\mathbb{E}(dS(t)) = rS(t)\,dt$.

From a corporate finance viewpoint, S is also equal to the present expected value of future dividends.

Assume for instance that the balance sheet of the corporate is very simple : there is a constant long term debt D and a total (liquid) asset $m(t)$. So that equity, $x(t)$ is equal to $m(t) - D$. The earnings $dB(t)$ of the corporate follow :

$$dB(t) = \mu dt + \sigma dW(t)$$

Where μ and σ are positive constants and W a standard Brownian motion.

Let $dL(t)$ the distributed dividends between t and $t + dt$, the variation of $x(t)$ (or $m(t)$) is equal to :

$$dx(t) = (\mu - rD)\,dt + \sigma dW(t) - dL(t)$$

The variation of equity is equal to ernings minus interests on the debt minus dividends.

Since D is constant the value S of the corporate only depends on x : there exists a real function $z \to V(z)$ such that

$$S(t) = V(x(t))$$

The purpose of this problem is to find V and the optimal strategy of dividend distribution!.

Assume that we are at a state where the level of equity is such that it is not optimal to distribute dividends : $dL(t) = 0$.

- Q1 Using Ito lemma write $dS(t)$ (remark that $V(.)$ does NOT depend on t)
- Q2 Using $\mathbb{E}(dS(t)) = rS(t)\,dt$, deduce a second order differential equation for the function V.
- Q3 Find the general solution for V. (I guess you know how to integrate a second order linear differential equation with constant coefficients, otherwise hop Wikipedia...or try a linear combination of exponentials).

To find the two unknown coefficients of the general solution we need 2 conditions.

First we assume that if x hits 0, the corporate goes to bankrupcy so that $V(0) = 0$: this gives one condition.

The second one will be found by considering dividend strategy. It can be shown first that V is a concave function. We admit that.

Assume the level of equity is x and consider taking h for dividends. The shareholder value after this operation is hence $V(x-h)+h$.

The optimal dividend strategy is obtained by solving :

$$\max_{x \geq h \geq 0} V(x-h)+h$$

or equivalently

$$\max_{0 \leq z \leq x} V(z) - z + e$$

- Show that it is not optimal to distribute dividends when the level of equity is such that $V'(x) > 1$.

Let x^* the value of x such that $V'(x^*) = 1$. The optimal dividend strategy consists in distributing dividends when $x = x^*$so that $dx(t) = 0$, and no dividend when $x < x^*$.

- Comment this strategy from a corporate finance viewpoint...

Fix x^*.

- Write $V'(x^*) = 1$: this gives a second condition.
- Give the expression of $V(x)$ function of x^* (by finding the the two coefficients of the general solution).
- The optimal x^* is obtained by maximizing this expression with respect to x^* : find it.