MODELES DE LA FINANCE

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Ce sujet comporte 4 pages, 2 exercices et un exercice en bonus! Chaque réponse ne nécessite qu'une ou deux lignes de raisonnement ou calcul

1 Long term savings product (30 minutes, 14 pts)

A financial institution wants to propose her clients a long term savings product. Consider a model with 3 dates dates 0 (today), 1 (short term) , 2 (long term). The product would be the following : for 1 euro saved at date 0, the client can exit the product (and take the cash away) at date 1 or 2 : if he gets out at date 1 he gets $1+\rho$ euros and if he gets out at date 2 he gets $(1+\rho)^2$. The problem is to fix ρ.

The characteristics of the bond market are the following.

- At date 0, two zero coupons are available :
  - the ZC(0,1), maturity 1 (gives 1 euro at date 1) whose price (at date 0) is $B(0,1) \equiv \frac{1}{1+r_{01}}$
  - the ZC(0,2), maturity 2 (gives 1 euro at date 2) whose price (at date 0) is $B(0,2) \equiv \frac{1}{(1+r_{02})^2}$

At date 0 one knows that at date 1, a new ZC(1,2) (that gives 1 euro at date 2) will be available. But its price is not known at date 0. It is assumed that there are 2 states of nature : in the state "high", the price is $B(1,2) = \frac{1}{1+R_{12}}$ i.e. the (one period) interest rate R_{12} (between date 1 and 2) is high, and in the state "low" $B(1,2) = \frac{1}{1+r_{12}}$ i.e the interest rate is low : $r_{12} < R_{12}$.

First we have to find the "risk neutral probability" (or equivalently the pseudo prices or arbitrage free weights)

- Consider two different (dynamic) portfolios :
  - Portfolio A : buy 1 unit of ZC(0,1) and reinvest the cash obtained at date 1 (i.e. 1 euro) on the new ZC(1,2).
  - Portfolio B : buy 1 unit of ZC(0,2)
- **Q1** *Write the vector price p (at date 0) and the matrix D of payoffs (at date 2) associated to these two portfolios.*

Let $q(h)$ and $q(\ell)$ the arbitrage free weights or pseudo prices (between date 0 and date 2).

- ***Q2*** *Write the two equations defining $q(h)$ and $q(\ell)$.*
- ***Q3*** *Compute $q(h)$ and $q(\ell)$. Under which condition arbitrage free hypothesis is verified.*
- ***Q4*** *What is the risk neutral probability distribution between 0 and 2.*

It will be convenient to note : $\frac{(1+r_{02})^2}{(1+r_{01})} \equiv 1 + f_{12}$ (f_{12} is the forward rate)

Consider now the long term saving product with $r_{12} < \rho < r_{01} < R_{12}$

Consider 1 euro saved at date 0 on this product.

- ***Q5*** *What is the optimal strategy (exit or remain) for the client at date 1 in each state of nature.*
- ***Q6*** *What are the (optimal) payments at date 2 of 1 euro saved at date 0 with this product.*
- ***Q7*** *Write (and solve) the second degree equation (in $1+\rho$) giving the value of ρ*

2 Perpetual american put (20 minutes, 6 pts)

A perpetual american put belongs to the class of assets for which it is (rather) easy to find a price formula.

An american perpetual put is "the right to sell a stock at a given price K at any date". The difference with an european put is twofold : there is no expiration date (this is why it is called perpetual), and it can be exercised at any time (this is why it is american). The assumptions are the following : there is at any time a risk-free asset whose instantaneous rate of return is r (constant) per unit of time, the underlying stock has a price that follows the risk neutral dynamics :

$$dS(t) = rS(t)dt + \sigma S(t)dW(t) \tag{1}$$

Given the definitions above, suppose you hold a perpetual american put . If you decide at a date $\widehat{t}$ to exercise it, you obtain at that date a cash flow equal to $K - S\left(\widehat{t}\right)$. So, the problem is the following : at a given date, observing $S\left(t\right)$, you have to take the decision wait or exercice. Intuitively, there must exist a threshold value $\underline{S}$ such that you must wait if $S(t) > \underline{S}$ and exercise at the first time S hits $\underline{S}$, so that you get $K - \underline{S}$.

The problem is to find this $\underline{S}$.

Let $P(t)$ the value of the put at time t. Obviously this value depends only on the value of the stock : $P\left(t\right) \equiv V\left(S\left(t\right)\right)$, where V is "the value function" we are trying to find.

Assume we are at a date where $S(t) > \underline{S}$. So that waiting is optimal.

- Q1 Write $dP\left(t\right)$ (using Ito lemma, and remarking that $V(.)$ does NOT depend on t).
- Q2 Why it is necessary to have (apologies to Sebastien!) : $\mathbb{E}\left(dP\left(t\right)\right) \equiv D_S V\left(\left(S\left(t\right)\right)\right)dt = rP\left(t\right)dt = rV\left(S\left(t\right)\right)dt$, where the expectation is "under the risk neutral dynamics" and D_S the Dynkin operator under the risk neutral dynamics.

This must be true for any time such that $S(t) > \underline{S}$. So that this equation is valid for any underlying value $x = S(t)$ larger than $\underline{S}$. Replacing $S(t)$ by x gives a second order differential equation for V.

- Q3 Write the second order differential equation followed by $x \to V\left(x\right)$.

This is a Riccati equation whose general solution is Ax^{α}

- Q4 Solve and find α_1 (the other solution α_0 is obvious and corresponds to the stock) such that $V_A\left(x\right) = Ax^{\alpha_1}$

For the moment we don't know A. To find A it is necessary to know the value at some point.

- Q5 What must be the value for $x = \underline{S}$? (you exercise the option)
- Q6 Give the expression of $V\left(S\right)$ with respect to $\underline{S}$
- Q7 What is the value $\underline{S}^*$ of $\underline{S}$ that maximizes $V\left(S\right)$, for any S

3 Bonus : Reserve of an insurance company? Or a problem of leaks in a bath, (30 minutes, many pts) . (Useful to obtain a letter of recommendation...)

Consider an insurance company whose market value (the value of the all the stocks) is $S\left(t\right)$. Assume there is a risk-free asset with constant yield per unit of time r. Under the risk neutral probability we have hence (with a strong abuse of notation) $\mathbb{E}\left(dS\left(t\right)\right) = rS\left(t\right)dt$.

From a corporate finance viewpoint, S is also equal to the present expected value of future dividends.

Assume for instance that the balance sheet of the insurance company is very simple : there is a constant long term debt D and a total (liquid) asset $m(t)$. So that equity, $x(t)$ is equal to $m(t) - D$.

The insurance company collects insurance premiums. Let $\Pi(t)$ the cumulated collected premiums. We assume $d\Pi(t) = \pi dt$ (π is the constant flow of premium per unit of time). The cumulated claims $L(t)$ (loss that the company is commited to indemnify) is random : $dL(t) = \ell dt + \sigma dW(t)$, where W is a standard brownian motion.

The earnings of the insurance company hence follow :

$$dB(t) = (\pi - \ell)\,dt - \sigma dW(t)$$

Let $dG(t)$ the distributed dividends between t and $t + dt$, the variation of $x(t)$ (or $m(t)$) is hence equal to :

$$dx(t) = (\pi - \ell - rD)\,dt - \sigma dW(t) - dG(t)$$

The variation of equity is equal to ernings minus interests on the debt minus dividends.

Since D is constant the value S of the corporate only depends on x : there exists a real function $z \to V(z)$ such that

$$\forall t, S(t) = V(x(t))$$

The purpose of this problem is to first find V, then the optimal reserve strategy and the optimal strategy of dividend distribution!

Assume that we are at a state where the level of equity is such that it is not optimal to distribute dividends : $dG(t) = 0$.

We note $\mu \equiv \pi - \ell - rD$

- Q1 Using Ito lemma write $dS(t)$ (remark that $V(.)$ does NOT depend on t)
- Q2 Using $\mathbb{E}(dS(t)) = rS(t)\,dt$, deduce a second order differential equation for the function $z \to V(z)$.
- Q3 Find the general solution for V. (I guess you know how to integrate a second order linear differential equation with constant coefficients, otherwise hop Wikipedia...or try a linear combination of exponentials).

To find the two unknown coefficients of the general solution we need 2 conditions.

It can be shown first that V is an increasing (obvious) concave function. We admit that.

Second we assume that if x hits 0, the insurance company must recapitalize. But this recapitalization (injecting cash by issuing new shares) is costly : to inject 1 euro, it costs $1 + \gamma$ euros to the shareholder.

Assume the level of equity is x. And consider injecting new cash h. The shareholder wealth after this operation is hence $V(x + h) - (1 + \gamma)h$. It would be optimal to inject an amount of cash :

$$\arg\max_{h \geq 0} V(x + h) - (1 + \gamma)h$$

The derivative of the maximand wrt h is equal to $V'(x + h) - (1 + \gamma)$

- Q4 Show that it is not optimal to recapitalize when $V'(x) \leq 1 + \gamma$
- Q5 Write $V'(0) = 1 + \gamma$. This gives one condition.

Assume the level of equity is x and consider taking h for dividends. The shareholder wealth after this operation is hence $V(x - h) + h$.

The optimal dividend strategy is hence obtained by solving :

$$\max_{x \geq h \geq 0} V(x - h) + h$$

or equivalently

$$\max_{0 \leq z \leq x} V(z) - z + x$$

- Q4 Show that it is not optimal to distribute dividends when the level of equity is such that $V'(x) > 1$.

Let x^* the positive value of x such that $V'(x^*) = 1$. The optimal dividend strategy consists in distributing dividends when $x = x^*$so that $dx(t) = 0$, and no dividend when $x < x^*$.

Fix x^*.

- Q5 Write $V'(x^*) = 1$: this gives a second condition.
- Q6 Give the expression of $V(x)$ function of x^* (by finding the the two coefficients of the general solution).
- Q7 The optimal x^* is obtained by maximizing this expression with respect to x^* : find it.
- Q8 What is the condition under which $V(0)$ is positive ?