RISK AVERSION

This chapter intends to develop the basic “expected utility assumption”.

1 The expected utility hypothesis

Get back to the Sempronius problem. His final wealth, when he trusts only one ship, can be described by a lottery $\widetilde{x}$. With probability 1/2, x is equal to 8000 ducats and with probability 1/2 it is equal to 4000 ducats. The mean of this lottery is 6000 ducats. Instead of trusting one boat, he now trusts equal proportion of his 4000 ducats in two ships which follow idependant but equally dangerous routes. This behaviour corresponds to the famous ”adage” according to which it is better not to put all eggs in the same basket !

In this situation what are the different possible events ? The two boats are safe, only one boat sinks, two boats sink. As the rouites are independant, the probability of these events are respectively : $1/2 \times 1/2 = 1/4$, $2 \times 1/2 \times 1/2 = 1/2, 1/2 \times 1/2 = 1/4$.

The lottery is then $\widetilde{y}$ where $y = 8000$ with probability 1/4, 6000 with probability 1/4, and 4000 with probability 1/4. This lottery has obviously the same expectation as the initial one. However, Sempronius prefers the second solution because he has the intuition that doing so he reduces the risk. Indeed if we compare the probability densities we motice that the second distribution is obtained by reducing the probabilities of ”extremal events” and increasing the probability of ”central one”.

Computing the expectation of the lottery is not a good measure of the ”value” since Sempronius prefers unambiguously the second one although they have the same expectation.

Because reducing risk is valuable, this means that, by reference to the mean value 6000, the negative effect -loosing 2000 ($4000 = 6000 - 2000$)- is not compensated, in the mind of Sempronius, by the positive one -gaining 2000 ($8000 = 6000 + 2000$) is . These two events (loose or gain 2000) have the same probability either with $\widetilde{x}$ or $\widetilde{y}$. But with $\widetilde{y}$ this probability is lower so that the global effect (which is negative since loosing is not compensated by gaining) is less harmfull.

This means that the value of a lottery is non linear with respect to wealth.

What other mesure than expactation can we hence propose. The idea is very simple. It is called the ”expected utility hypothesis”.

The hypothesis we make is that a decision maker evaluates a lottery on his wealth through its expected utility. Instead of computing the expectation of monetary outcomes, individuals use the expectation of some function (utility) of this wealth. In other words, people does not extract welfare directly from wealth, they rather extract utility from goods that can be purchased with this wealth. The expected utility hypothesis says that there is a non linear relationship between wealth and the utility of comsuming goods that are affordable with this wealth. In mathematical words, a decision maker uses an increasing fuction u to compare lotteries. To compare $\widetilde{y}$ and $\widetilde{x}$, he compares $\mathrm{E}(u(\widetilde{x}))$ and $\mathrm{E}(u(\widetilde{y}))$.

Definition 1. *expected utility (EU)assumption. To compare two lotteries on wealth, a decision maker uses a continuous increasing function u. $\widetilde{x}$ is prefered to $\widetilde{y}$ if and only if $E(u(\widetilde{x})) \geq E(u(\widetilde{y}))$.*

This hypothesis can be rationalized through an axiomatic approach (which defines a set of assumptions on behaviour that lead to the expected utility model). [1]. This axiomatization approach is beyon the aim of this courses. The interseted readers can refer to the first chapter of Gollier's book "The Economics of Risk and Time".

In this course we deliberately use the EU model.

2 The preference for diversification

What does preference for diversification means ? Suppose, as we have assumed in the previous section, that the value of a lottery is given by the expectation of some function u of its payments.

Let $\widetilde{\delta}$ the basic "Bernoulli" variable : with probability p, $\widetilde{\delta}_p$ is equal to 1 and with probability $1-p$ it is equal to 0. Take two such independant variables $\widetilde{\delta}_1$ and $\widetilde{\delta}_2$. In some sense the $\widetilde{\delta}_i$ is a variable indicating the result of a very simple lottery. In the example above, $\widetilde{\delta}_i$ is equal to 1 if the boat arrives to the port, 0 if it sunks on her trip. The decision maker can put his whole wealth in one boat (his total wealth will be $w + x\widetilde{\delta}_1$) or share it in two boats (the total wealth will be $w + \frac{x}{2}\widetilde{\delta}_1 + \frac{x}{2}\widetilde{\delta}_2$).

If the decision maker finds more valuable to share, we say he has preference for diversification.

Definition 2. *The utility function u exhibits preference for diversification if, for all sure wealth w, for all level x of risky wealth the decision maker pefers splitting the risky wealth :*

$$\mathbb{E}\left[u\left(w + \frac{x}{2}\widetilde{\delta}_1 + \frac{x}{2}\widetilde{\delta}_2\right)\right] \geq \mathbb{E}\left[u\left(w + x\widetilde{\delta}_1\right)\right]$$

Developping expectations gives :

$$\begin{aligned} & p^2 u(w+x) + 2p(1-p)u\left(w + \frac{x}{2}\right) + (1-p)^2 u(w) \\ \geq \quad & pu(w+x) + (1-p)u(w) \end{aligned}$$

As this must hold for all p, this gives :

$$u\left(w + \frac{x}{2}\right) - u(w) \geq u(w+x) - u\left(w + \frac{x}{2}\right)$$

This inequality shows that the loss of $\frac{x}{2}$ has not the same weight on "utility" according to the level of initial wealth. It is more painful to loose them than it is beneficial to gain them.

It is easy to see that preference for diversification is linked to the concavity of the utility function u.

Proposition 3. *The utility function u exhibits preference for diversification, if and only if u is concave :*

$\forall\ x, y \in \mathbb{R}, \forall \alpha \in [0,1]$,
$u(\alpha x + (1-\alpha)y) \geq \alpha u(x) + (1-\alpha)u(y)$

1. This has been done by Von Neumann and Morgenstern and others who have proved that the expected utility model is the one that satisfies an axiom of independence.

3 Risk aversion

Definition 4. *We say that a decision maker is risk-averse if and only if he dislikes zero mean lotteries, that is he does not want to take a risk with zero mean :* $\forall \widetilde{x}$, *s.t.* $E(\widetilde{x}) = 0, \forall w\ E[u(w + \widetilde{x})] \leq u(w)$

Remark 5. *This obviously is equivalent to say that he does not want to take any risk with negative mean : indeed, if* $\widetilde{z}$ *is a lottery with negative mean, then we have :* $\mathbb{E}\left[u(w + \widetilde{z})\right] = \mathbb{E}u(w+\mathbb{E}(\widetilde{z})+\widetilde{z}-\mathbb{E}(\widetilde{z}))$
$\leq u(w + \mathbb{E}(\widetilde{z})) \leq u(w)$.

A risk averse decision maker will hence accept to take a risk if the mean of the lottery is sufficiently large and positive. Similarly, a risk averse individual is ready to pay to avoid a zero mean risk. The maximum amount he is ready to pay is called the risk-premium.

Definition 6. *Given a zero mean risk,* $\widetilde{x}$*, the risk premium* $\Pi_u(\widetilde{x})$ *for a risk averse individual is defined by :*

$$\mathbb{E}\left[u(w + \widetilde{x})\right] = u(w - \Pi_u(\widetilde{x}))$$

Risk aversion is linked to the concavity of u.

Proposition 7. *A decision maker is risk averse if and only if* u *is concave..*

Démonstration. This can be easily shown by writing that the definition of risk aversion is equivallent to $\forall \widetilde{y}$, $\mathbb{E}\left[u(\widetilde{y})\right] \leq u(\mathbb{E}(\widetilde{y}))$ □

Risk aversion and preference for diversification are hence equivallent in the framework of expected utility model.

4 The measure of risk aversion

The pupose of this section is to try to give a consistent definition to "more" or "less" risk aversion.

4.1 degrees of risk aversion

When can we say that a decision maker is "more risk averse" than another one ? We know that a necessary condition for a risk to be accepted by a risk averse decision maker is that it has a strictly positive mean. A natural definition could then be :

Definition 8. *An invividual (entailed with a utility function* v*) is more risk averse than an individual (with a function* u*) if and only if, whenever* u *refuses a risk* $\widetilde{z}$*, then* v *refuses too.*
$\forall z$, $\mathbb{E}(u(w + \widetilde{z})) < u(w) \Longrightarrow \mathbb{E}(v(w + \widetilde{z})) < v(w)$

Take two such individuals. Take the real (increasing) function ϕ defined on the image of u (that is on $u(\mathbb{R})$), by $\phi = v \circ u^{-1}$. Take any lottery z , and set $\widetilde{y} = u(w + \widetilde{z})$ we have : $\mathbb{E}v(w + \widetilde{z})) = \mathbb{E}\left[\phi(u(w + \widetilde{z}))\right] = \mathbb{E}\left[\phi(\widetilde{y})\right]$

Suppose $\mathbb{E}(u(w+\widetilde{z})) < u(w)$, that is $\mathbb{E}(\widetilde{y}) < u(w)$, Hence, because v is more risk averse, $\mathbb{E}\left[\phi(\widetilde{y})\right] < \phi(u(w))$

Hence the definition amounts to say : $\forall \widetilde{y}$ and c such that $\mathbb{E}(\widetilde{y}) < c$, $\mathbb{E}\left[\phi(\widetilde{y})\right] < \phi(c)$

which implies $\forall \widetilde{y}\ \mathbb{E}\left[\phi(\widetilde{y})\right] \leq \phi(\mathbb{E}\widetilde{y}))$, .which means that ϕ is concave (if it were not the case there would exist $\widetilde{y}_0$ such that $\mathbb{E}\left[\phi(\widetilde{y}_0)\right] > \phi(\mathbb{E}(\widetilde{y}_0))$. and then c, such that $\phi(\mathbb{E}(\widetilde{y}_0)) < \phi(c) < \mathbb{E}\left[\phi(\widetilde{y}_0)\right])$

Conversely, if $\phi = v \circ u^{-1}$ is concave, it is easy to see that v is more risk-averse tha u.

Proposition 9. *An invividual (entailed with a utility function v) is more risk averse than an individual (with a function u) if and only if v is a concave transformation of u : $v = \phi \circ u$, with ϕ concave.*

It is *then* easy to see that the risk premium of a more risk averse decision maker is larger that the one of a less one, that is $\Pi_v(\widetilde{x}) \geq \Pi_u(\widetilde{x})$ for all zero mean risk $\widetilde{x}$. Indeed, as $v = \phi \circ u$, we have $v(w - \Pi_v(\widetilde{x})) = \phi \circ u(w - \Pi_v(\widetilde{x})) = \mathbb{E}\left[\phi \circ u(w+\widetilde{x})\right] \leq \phi\left[\mathbb{E}(u(w+\widetilde{x}))\right] = \phi(u(w - \Pi_u(\widetilde{x})) = v(w - \Pi_u(\widetilde{x}))$

4.2 differentiable case and index of absolute risk aversion

Suppose that u and v are twice continuously differentiable. In such a case concavity of u and v are equivalent to say that v" and u" are negative functions. A direct calculus gives : $v' = (\phi' \circ u)u'$ and $v'' = (\phi'' \circ u)(u')^2 + (\phi' \circ u)u''$

that is $\frac{-v''}{v'} = \frac{-u''}{u'} + \frac{-(\phi'' \circ u)}{(\phi' \circ u)}u'$. As (by concavity) $\phi'' \leq 0$, $\frac{-v''}{v'} \geq \frac{-u''}{u'}$. This motivates the following definition.

Definition 10. *For a decision maker entailed with a utility function u, the Absolute Risk Aversion Index $I_u(w)$ at the level of wealth w is defined by $I_u(w) = \frac{-u''(w)}{u'(w)}$*

Proposition 11. *The following properties are equivallent*

v is more risk averse than u

(ii) $v = \phi \circ u$, with ϕ concave

(iii) $\Pi_v(\widetilde{x}) \geq \Pi_u(\widetilde{x})$ for all zero mean risk $\widetilde{x}$

(iv) $\frac{-u''(w)}{u'(w)} \leq \frac{-v''(w)}{v'(w)}$, that is $I_u(w) \leq I_v(w)$ for all w, if we restrict to twice continuously differentiable functions.

4.3 Small risks and Arrow-Pratt approximation

It is interesting to examine the behaviour of risk premium for small risk and twice continuously differentiable functions u. Fix a zero mean lottery $\widetilde{x}$ and set $\widetilde{y} = k\widetilde{x}$. The risk premium associated with $\widetilde{y}$ is defined by $E\left[u(w+\widetilde{y})\right] = u(w - \Pi_u(\widetilde{y}))$. For $\widetilde{x}$ fixed we can examine the behaviour of the risk premium for small values of k. Set $g(k) = \Pi_u(\widetilde{y}) = \Pi_u(k\widetilde{x})$. We have $g(k) = g(0) + kg'(0) + \frac{k^2}{2}g''(0) + o(k^2)$.

Obviously $g(0) = 0$: when there is no risk there is no risk premium!. To compute $g'(0)$, we use an implicit function argument. Consider the identity $\mathbb{E}\left[u(w+k\widetilde{x})\right] = u(w - g(k))$ which is true (by definition of the risk premium) for all k. Differentiating both sides with respect to k gives : $\mathbb{E}(\widetilde{x}u'(w+k\widetilde{x})) = -g'(k)u'(w-g(k))$. For $k=0$ this gives $\mathbb{E}(\widetilde{x}) = -g'(0)$. As $\mathbb{E}(\widetilde{x}) = 0$, this implies $g'(0) = 0$. Risk aversion is a second order phenomenon (for continuously differentiable functions). The risk premium is nul at a first order approximation. Differentiating twice gives $\mathbb{E}(\widetilde{x}^2 u''(w+k\widetilde{x})) = -g''(k)u'(w-g(k)) - (g'(k))^2 u''(w-g(k))$. For $k=0$ this gives $g''(0) = \frac{-u''(w)}{u'(w)}\mathbb{E}(\widetilde{x}^2) = I_u(w)\mathbb{E}(\widetilde{x}^2)$

Proposition 12. *For a twice continuously differntiable function u, the Arrow Pratt approximation of the risk premium is :*
$\Pi_u(\widetilde{y}) \sim \frac{1}{2} I_u(w)\ var(\widetilde{y}) + o(var(\widetilde{y}))$

5 Decreasing risk aversion

We are now interested in determining how the risk premium is affected by a change in initial wealth w. The intuition, and some empirical evidences, seems to imply that wealthier people bear more easily risk than poorer. The risk premium for a given zero mean risk is decreasing with wealth.

For all given w and zero mean risk $\widetilde{x}$, The risk premium $\Pi_u(\widetilde{x}, w)$ verifies :

$$\mathbb{E}(u(w + \widetilde{x})) = u(w - \Pi(\widetilde{x}, w))$$

Differentiating with respect to w gives :

$$\mathbb{E}(u'(w + \widetilde{x})) = u'(w - \Pi_u(\widetilde{x}, w))(1 - \Pi'_w(\widetilde{x}, w))$$

$$\Pi'_w(\widetilde{x}, w)u'(w - \Pi_u(\widetilde{x}, w)) = u'(w - \Pi_u(\widetilde{x}, w)) - E(u'(w + \widetilde{x}))$$

This means that the risk premium is decreasing with w if (for all $\widetilde{x}$ and w) :

$$\mathbb{E}(-u'(w + \widetilde{x})) \leq -u'(w - \Pi_u(\widetilde{x}, w))$$

As the function $-u'$ is increasing, this implies that $-u'$ is concave, i.e u' convex.. Indeed $-u'(w - \Pi_u(\widetilde{x}, w)) \leq -u'(w) = -u'(E(w + x))$.

Definition 13. *We say that a risk-averse decision maker is "prudent" if u' is a convex decreasing function.*

This definition is somewhat mysterious, but we shall see in the sequel that this indeed corresponds to a prudent behaviour.

Proposition 14. *Only prudent decision makers have decreasing risk premium with initial wealth.*

Decreasing risk aversion implies however more. Indeed $\mathbb{E} - u'(w + \widetilde{x})) \leq -u'(w - \Pi_u(\widetilde{x}, w))$ means that the risk premium associated with the "utility function" $v = -u'$ is higher than the one for u. That means that $-u'$ is more concave than u.

With the previous propositions this is equivalent to have $\frac{-u'''}{u''} \geq \frac{-u''}{u'}$.

Take $I_u(w) = \frac{-u''(w)}{u'(w)}$. $I'_u(w) = \frac{-u'''u' + u''^2}{u'^2}$. As u' is positive, $I'_u(w) \leq 0$ if and only if $\frac{-u'''}{u''} \geq \frac{-u''}{u'}$. We have hence the following proposition :

Proposition 15. *The risk premium is decreasing with wealth if and only if the index of absolute risk aversion is decreasing with wealth.*

5.1 Aversion for downside risk

Consider a variation of the problem of Sempronius. Suppose that the value itself of x (the foreign wealth) are not sure. For instance the prices are subject to variations due to changes in demand. The lottery $\widetilde{y}_1$ is then defined in the following way : with probability $\frac{1}{2}$ the boat perishes and the cargo is lost , $y_1 = w$ and with probability $\frac{1}{2}$, when the boat succeeds, the wealth is (with equal probability) either $w + x- \epsilon$ or $w + x + \epsilon$.

We have obviously

$$\mathbb{E}(\widetilde{y}_1) = \frac{1}{2}w + \frac{1}{2}(\frac{1}{2}(w + x - \epsilon) + \frac{1}{2}(w + x + \epsilon)) = w + \frac{1}{2}x$$

Suppose now that the uncertainty (the noise) affects the bad state of nature. The lottery $\widetilde{y}_2$, is such that with probability $\frac{1}{2}$ the wealth is ether (with equal probability) $w-\epsilon$ or $w+\epsilon$., and with probability $\frac{1}{2}$ it is equal to $w + x$. We have also $\mathbb{E}(\widetilde{y}_2) = w + \frac{1}{2}x$.

It is easy to see that the variances are also identical : $var(\widetilde{y}_1) = var(\widetilde{y}_2)$.

Th intution suggests that Sempronius would prefer the first one : he dislikes "downside" risk, that is risk beared by the bad states of nature.

We have

$$\begin{aligned} \mathbb{E}(u(\widetilde{y}_1) - \mathbb{E}(u(\widetilde{y}_2) &= \frac{1}{2}\left(u(w) - \mathbb{E}\left(u(w + \widetilde{\epsilon})\right)\right) + \frac{1}{2}\left(\mathbb{E}\left(u(w + x + \widetilde{\epsilon})\right) - u(w + x)\right) \\ &= \frac{1}{2}\int_{w}^{w+x} (u'(s) - \mathbb{E}(u'(s + \widetilde{\epsilon}))ds \end{aligned}$$

$\mathbb{E}(u(\widetilde{y}_1) \geq \mathbb{E}(u(\widetilde{y}_2)$ for all x and w implies that for all s and all ϵ , $u'(s) \geq \frac{u'(s-\epsilon)+u'(s+\epsilon)}{2}$. With the same proof as for proposition 1, this is equivalent to u' convex.

Proposition 16. *A decision maker is averse for downside risk if and only if u' is a decreasing positive and convex function.*

Prudence and downside risk aversion are equivallent concepts.

6 Classical utility functions

In this section we give some families of utility functions that are commonly used in Economics and Finance. Obviuouly, assuming that the decision maker has a specific utility function is rather restrictive. This is done to obtain tractable solutions to many problems. But we have to keep in mind that some of these are closely related to the choice of a narrow class of utility functions.

6.1 Quadratic

The first family is the quadratic set :

Definition 17. *Quadratic function : $u(w) = w - \frac{1}{2a}w^2$*

This function gives for a lottery $\widetilde{z}$ $\mathbb{E}\left(u(\widetilde{z})\right) = \mathbb{E}(\widetilde{z}) - \frac{1}{2a}\mathbb{E}(\widetilde{z}^2) = \mathbb{E}(\widetilde{z}) - \frac{1}{2a}\left(var(\widetilde{z}) + \mathbb{E}(\widetilde{z})^2\right)$

That is : $\mathbb{E}\left(u(\widetilde{z})\right) = u(\mathbb{E}\widetilde{z})) - \frac{1}{2a}var(\widetilde{z})$, which amounts to mean-variance models : for two lotteries having the same mean the decision maker will choose the one with the smaller variance.

The main drawback of this kind of utility function is the fact that it does not fulfill the decreasing risk aversion hypothesis.

Indeed :

$$I(w) = \frac{1}{a-w}$$

The eversion index is increasing with wealth. For this main reason, quadratic functions are no more used to model decision behaviour in front of risk.

6.2 CARA

CARA (Constant absolute risk-aversion) function are those for which $I_u(w)$ is constant.

Definition 18. *The family of CARA functions is the set of exponential function :*
$u(w) = -\frac{1}{\rho}\exp(-\rho w)$, *with* $\rho \geq 0$.
The index of absolute risk aversion is $I_u(w) = \rho$, *constant.*

This function is largely used for several reasons. One is very interesting. When the lottery $\widetilde{z}$ is normally distributed with mean m and a variance σ^2, we have (proof let to the reader) :

$$\mathbb{E}(u(\widetilde{z})) = u(m - \frac{1}{2}\rho\sigma^2)$$

That is to say that the risk premium is exactly $\frac{1}{2}\rho\sigma^2$. The Arrow-Pratt Approximation is exact.

6.3 Other harmonic absolute risk aversion functions.

We can easily obtain decrasing absolute risk aversion functions by taking $I_u(w) = \frac{\beta}{w}$. for some $\beta \geq 0$. If we call wI_u "the relative index of risk aversion" These functions are such that their relative index are constant. By doing so we define "Harmonic" Risk aversion functions. It is easy to see that that gives the following family.

Definition 19. *Constant relative risk aversion (CRRA) functions are defined by :*
$u(w) = \frac{1}{1-\beta}w^{1-\beta}$ *for* $\beta \neq 1$ *, and* $u(w) = \ln(w)$, *for* $\beta = 1$.

7 APPENDIX

Démonstration. [Proof of Proposition 1] The utility function u exhibits preference for diversification, if and only if u is concave.

preference for diversification means :
$\forall w, x : u\left(w + \frac{x}{2}\right) - u(w) \geq u(w+x) - u\left(w + \frac{x}{2}\right)$
that is :
$\forall x, y : u\left(\frac{x+y}{2}\right) \geq \frac{u(x)+u(y)}{2}$
take then $\alpha \in [0,1]$, it can be written in dyadic form :

$\alpha = \sum_{k=1}^{\infty} \frac{\varepsilon_k}{2^k}$, $1 - \alpha = \sum_{k=1}^{\infty} \frac{1-\varepsilon_k}{2^k}$
We have :
$u(\alpha x + (1-\alpha)y) = u\left(\sum_{k=1}^{\infty} \frac{\varepsilon_k x + (1-\varepsilon_k)y}{2^k}\right)$
that is :
$u(\alpha x + (1-\alpha)y = u\left(\frac{\varepsilon_1 x + (1-\varepsilon_1)y + \sum_{k=2}^{\infty} \frac{\varepsilon_k x + (1-\varepsilon_k)y}{2^{k-1}}}{2}\right)$
preference for diversification implies :
$u\left(\frac{\varepsilon_1 x + (1-\varepsilon_1)y + \sum_{k=2}^{\infty} \frac{\varepsilon_k x + (1-\varepsilon_k)y}{2^{k-1}}}{2}\right) \geq \frac{u(\varepsilon_1 x + (1-\varepsilon_1)y) + u\left(\sum_{k=2}^{\infty} \frac{\varepsilon_k x + (1-\varepsilon_k)y}{2^{k-1}}\right)}{2}$
as ε_1 is either 0 or 1 $u\left(\varepsilon_1 x + (1-\varepsilon_1)y\right) = \varepsilon_1 u(x) + (1-\varepsilon_1)u(y)$
Then we have :
$u(\alpha x + (1-\alpha)y \geq \frac{\varepsilon_1}{2}u(x) + \frac{1-\varepsilon_1}{2}u(y) + \frac{1}{2}u\left(\sum_{k=2}^{\infty} \frac{\varepsilon_k x + (1-\varepsilon_k)y}{2^{k-1}}\right)$
doing the same trick with $u\left(\sum_{k=2}^{\infty} \frac{\varepsilon_k x + (1-\varepsilon_k)y}{2^{k-1}}\right) = u((2\alpha - \varepsilon_1)x + (1 - \varepsilon_1 - 2\alpha)y)$, (and so on) gives the result. □