1 Discrete

1.1 Cox-Ross-Rubinstein model with 3 dates

Consider the (generalized) Cox-Ross-Rubinstein model with 3 dates 0,1,2. At date 1 : two states of nature $\omega_1 = (u)$ or (d) , at date 2 : $\omega_2 = (uu)$ or (ud) (successors of (u),or $\omega_2 = (du)$ or (dd) (successors of (d)). The process of the stock is given :

$S_0,\ S_1(u), S_1(d), S_2(uu), S_2(ud), S_2(du), S_2(dd)$. Notice that we don't assume geometrical random walk (i;e multiplying the price by some factors).

At any date there is a locally risk free asset (short term bond).

The process of (short term) interest rates is given : $r_0\ r_1(u)$,$r_1(d)$

- Q0 What is the tree assoiated to the process
- Q0bis Write the equations that say that there is no arbitrage (unknown : risk neutral probability distributions ie $p(u/\omega_0), p(d/\omega_0) = 1 - p(u/\omega_0)$, $p(uu/u), p(ud/d) = 1 - p(uu/u)$, $p(du/d), p(dd/d) = 1 - p(du/d)$)
- Q1 give the values of the risk-neutral probabilities of transition at date 0 and at date 1 in each state, functions of $S_i(\omega_i)$ and $r_i(\omega_i)$.
- Q2 What are the values of the zero coupon bond terminal time $T = 2$, at dates 0 and 1.
- Q3 What is the price of the "option" giving 1 euro in states uu or dd and zero elsewhere.

1.2 Exotic american call

Consider the simplified Cox-Ross-Rubinstein model on a tree with 3 dates 0,1,2. At date 1 : two states of nature (u) and (d) , at date 2 : (uu) (ud) (du) (dd). There is a risk-free asset with constant one period return equal to r. There is a Stock whose process is given :

$S_0, S_1(u) = uS_0, S_1(d) = dS_0, S_2(uu) = u^2S_0, S_2(ud) = S_2(du) = udS_0, S_2(dd) = d^2S_0$.

u and d are positive constants such that : $u > 1 + r > d > 0$

- Q1 Recall the arbitrage free condition, give the two equations that define the transition weights $q(u/\omega_0) = q(uu/u) = q(du/d) \equiv q_u$ and $q(d/\omega_0) = q(dd/d) = q(ud/u) \equiv q_d$.

For the sequel it could be more convenient to use these two equations instead of the explicit expression of state prices q_u and q_d...

We define an exotic american call in the following way. One can exercice it at any date t (0,1or 2) at a price equal to $\alpha^t S_0$ (i.e. at date 0 the exercise price is S_0 at date 1, αS_0 and at date 2, $\alpha^2 S_0$) where α is a positive constant. We make the following assumption :

$u > \alpha > 1 + r$ and $ud > \alpha^2$. To find the value of such a call at date 0 we follow a backward analysis : find the value at date 2 in each state and deduce the price at date 1 in each state and deduce the value at date 0.

- Q2 What are the values of the call at date 2 in each of the 4 states (this value is simply the payoff)?
- Q3 Consider the call at date 1 in the state (u). One can keep it of exercise it. If it were optimal to keep it, what would be its (resell) value at that date? What is the payoff if one exercises it?
- Q4 Deduce the optimal strategy and the value of the call in that state.
- Q5 Do the same reasonning in the state (d).
- Q6 Give the value at date 0

2 Continuous

2.1 Ito lemma (simple)

Consider a process X (where a,b and σ are positive real numbers):

$$dX(t) = (aX(t) - b)\,dt + \sigma dW(t)$$

Take the deterministic function $X_0(t) \equiv \frac{b}{a}(1 - \exp(at))$

- Q1 Compute dX_0

Set f: $(x,t) \to \exp(-at)(x - X_0(t)) = \exp(-at)\,x + \frac{b}{a}(1 - \exp(-at))$

- Q2 Compute $\frac{\partial f}{\partial t}(x,t)$, $\frac{\partial f}{\partial x}(x,t)$, $\frac{\partial^2 f}{\partial x^2}(x,t)$
- Q3 Take $Y(t) = f(X(t), t)$ and apply Ito lemma to show $dY(t) = \exp(-at)\,\sigma^2 dW(t)$

$Y$is hence a martingale. We can rewrite, integrating dY on $(0,t)$:

$$X(0) = \exp(-at)\,X(t) + \frac{b}{a}(1 - \exp(-at)) - \sigma^2 \int_0^t \exp(-as)\,dW(s)$$

- Interpret for $a = r$ is the risk-free rate and bdt is a constant flow of dividend

2.2 Perpetual american put

A perpetual american put belongs to the class of assets for which it is (rather) easy to find a price formula.

An american perpetual put is "the right to sell a stock at a given price K at any date". The difference with an european put is twofold : there is no expiration date (this is why it is called perpetual), and it can be exercised at any time (this is why it is american). The assumptions are the following :

- There is at any time a risk-free asset whose instantaneous yield is r per unit of time (One euro invested gives $1 + rdt$ at time $t + dt$) ,r is fixed given.
- The underlying Stock has a price that follows a diffusion process :

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t) \tag{1}$$

where $B(t)$ is a standard brownian motion, σ is a fixed real number.

- So that (see course) the risk-neutral dynamics (the probability stucture such that the value of any asset is equal to the expected present value of its cash flows) is :

$$dS(t) = rS(t)dt + \sigma S(t)dB(t) \tag{2}$$

Given the definitions above, suppose you hold a perpetual american put . If you decide at a date $\widehat{t}$ to exercise it, you obtain at that date a cash flow equal to $K - S\left(\widehat{t}\right)$. So, the problem is the following : at a given date, observing $S\left(t\right)$, you have to take the decision wait or exercice. Intuitively, there must exist a threshold value $\underline{S}$ such that you must wait if $S(t) > \underline{S}$ and exercise at the first time S reaches $\underline{S}$, so that you get $K - \underline{S}$.

The problem is to find this $\underline{S}$.

Let P_t the value of the put at time t. Obviously this value depends only on the value of the stock : $P_t = V\left(S\left(t\right)\right)$, where V is "the value function" we are trying to find.

Assume we are at a date where $S(t) > \underline{S}$. So that waiting is optimal.

- Q1 Write dP_t (using Ito lemma).
- Q2 Why it is necessary to have (apologies to Etienne!) : $\mathbb{E}\left(dP_t\right) \equiv DP_t dt = rdtP_t$, where the expectation is "under the risk neutral dynamics" and D the Dynkin operator under the risk neutral dynamics.

This must be true for any time such that $S(t) > \underline{S}$. So that this equation is valid for any underlying value $x = S(t)$ larger than $\underline{S}$. Replacing $S(t)$ by x gives a second order differential equation.

- Q3 Write the second order differential equation followed by $x \to V\left(x\right)$.

This is a Riccati equation whose general solution is Ax^{α}

- Q4 Solve and find α_1 (the other solution α_0 is obvious and corresponds to the stock) such that $V_A(x) = Ax^{\alpha_1}$

For the moment we don't know A. To find A it is necessary to know the value at some point.

- Q5 What must be the value for $x = \underline{S}$? (you exercise the option)
- Q6 Give the expression of $V(S)$ with respect to $\underline{S}$
- Q7 What is the value S^* of $\underline{S}$ that maximizes $V(S)$, for any S
- Q8 Conclude : value and optimal exercise